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# A population equation with diffusion

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**Abstract**

In this paper we discuss a population equation with diffusion. It is different from the equation proposed, for example, in [K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, 2000] or in [J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, 1996] so far as it combines diffusion with delay. We explain the origin of this equation and study it with the theory developed by G. Fragnelli and G. Nickel (Differential Integral Equations 16 (2003) 327–348) and G. Fragnelli (Abstract Appl. Anal., in press).


## 1. Introduction

The aim of this paper is to analyse the following evolution equation:

$$\frac{\partial u}{\partial t}(t,x) = \Delta_N u(t,x) - du(t,x) + \int_{-r}^{0} f(x)b(-s)e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\Delta_D - d)} u(t+s,x)\,ds, \tag{1.1}$$

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where $u(t,x)$ represents the population density at time $t \geqslant 0$ and position $x \in \Omega \subseteq \mathbb{R}^n$, with the initial condition

$$u(0,x) = u_0(x).$$

Here $\Delta_N$ and $\Delta_D$ are the Laplacian operators (with respect to the spatial variable) with Neumann and Dirichlet conditions, respectively, and $e^{-s(\Delta_D - d)}$ denotes the strongly continuous semigroup generated by $\Delta_D - d$ on an appropriate Banach space. The constant $d$ and the function $b$ defined in $[0,r]$ with values in $\mathbb{R}_+$ represent the mortality rate and the birth rate while $f$ is the rate of fecundation. Finally, $r$ is the delay due to pregnancy.

The meaning of the previous equation is that the variation of the population density at time $t$ and position $x$ is given by the diffusion, i.e., by the migration of the population, minus the contribution due to deaths, plus the contribution due to births (depending on the delay). In particular, the last term of (1.1) takes account of the fact that the pregnant individuals move during the time between $t - r$ and $t$. Moreover, they are submitted to other processes that we will explain in the following.

An equation like (1.1) is studied in [8], where the authors analyze the global behaviour of a vector disease model which involves spatial spread and hereditary effects. Using contracting convex techniques they prove that, if the recovery rate is less than or equal to a threshold value, the disease dies out, while otherwise the infectious people density tends to a homogeneous distribution (see [12] for a model including predators and preys).

This paper is organized as follows.

In Section 2 we explain how to derive Eq. (1.1). Section 3 is divided into two subsections. In the first one we study the equation with the semigroup technique and to this aim we rewrite it as a delay equation with nonautonomous past (see [5] and [6]). In the second subsection we discuss its well-posedness. In the last section we find stability conditions for the solutions of (1.1).

## 2. Derivation of the equation

We consider a spatially distributed population where the individual state is characterized by the position, i.e., individuals are supposed to be equal except for the position $x$ they occupy. In particular, no sex or age differences are allowed. A special attention is paid to a mechanism, such as pregnancy, that leads to a delay in the replacement of the population. Therefore, within the total population we distinguish pregnant individuals and, in spite of the lack of age-structure, referring to them we consider the 'age of gestation' $a$, ranging in $[0,r]$ where $r > 0$ is fixed. Individuals are supposed to die at a given death rate $d$, to be fecundated at a rate $f(x)$ and to bear according to a rate $b = b(a)$. Moreover, as a first approximation, we assume that the dispersal of the population through the environment is realized by the Laplace operator. Hence, summing up, if $u(t,x)$ and $v(t,a,x)$ denote, respectively, the total population at time $t$ and position $x$ and the subpopulation collecting pregnant individuals that at time $t$ are at the position $x$ and have time of gestation $a$, the dynamics of the two populations is governed by the following equations:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \Delta_N u(t,x) - du(t,x) + \int_0^r b(a)v(t,a,x)\,da, \\ \frac{\partial v(t,a,x)}{\partial t} = -\frac{\partial v(t,a,x)}{\partial a} + \Delta_D v(t,a,x) - dv(t,a,x) - b(a)v(t,a,x), \\ v(t,0,x) = f(x)u(t,x), \end{cases} \tag{2.1}$$

where $\Delta_D$ and $\Delta_N$ denote the Laplace operators with Dirichlet and Neumann conditions, respectively, on a Banach space $X$ (see below). Here $t \geqslant 0$ and the space variable $x$ is supposed to vary in $\Omega \subseteq \mathbb{R}^n$ where $\Omega$ is open, connected and bounded with smooth boundary.

The condition

$$\frac{\partial u}{\partial n}(t,x) = 0 \quad \text{in } \partial\Omega, \tag{2.2}$$

where $n$ denotes, as usual, the outward normal, states that the population cannot cross the boundary.

Finally, the condition

$$v(t,a,x) \equiv 0 \quad \text{if } x \in \partial\Omega \tag{2.3}$$

says that no pregnant individual reaches the borderline. Therefore, $f(x) = 0$ for $x \in \partial\Omega$ is required.

Equations (2.1)–(2.3) together with the initial conditions

$$v(0,a,x) = v_0(a,x), \quad u(0,x) = u_0(x), \quad u(s,\cdot) = g(s,\cdot), \tag{2.4}$$

for all $s \in [-r,0]$ giving rise to a system of linear partial differential equations with initial and boundary conditions. Here $v_0$ and $u_0$ are given functions and $g \in L^1([-r,0],X)$ describes the prehistory of the system. Moreover $g(0,\cdot) = u_0(\cdot)$ is required. To analyse this system we start by solving the second equation along the characteristic lines in the plane $(t,a)$, namely in the strip $[0,+\infty) \times [0,r]$. Set

$$V(s,x) := v(t_0+s, a_0+s, x), \tag{2.5}$$

where $a_0 \in [0,r]$, $t_0 \geqslant 0$ are fixed, while $x$ varies in $\Omega$ and $s$ in $[0,+\infty)$. Rewriting the second equation in (2.1) for $V(s,x)$ one gets

$$\frac{\partial V}{\partial s}(s,x) = \Delta_D V(s,x) - dV(s,x) - b(a_0+s)V(s,x). \tag{2.6}$$

To solve (2.6) we follow the abstract approach choosing $X = L^1(\Omega)$ as a Banach space and we denote by $D(\Delta_D)$ the domain of $\Delta_D$ on $X$.

For $t < a$, putting $t_0 = 0$ in (2.5), we obtain

$$\begin{cases} V'(s) = \Delta_D V(s) - dV(s) - b(a_0+s)V(s), \\ V(0) = v(0,a_0,\cdot) = v_0(a_0,\cdot), \end{cases} \tag{2.7}$$

which has the unique solution

$$V(s) = e^{-\int_0^s b(a_0+\sigma)\,d\sigma} e^{s(\Delta_D - d)} V(0), \tag{2.8}$$

where $d$ denotes the multiplication operator and $e^{t(\Delta_D - d)}$ the strongly continuous semigroup generated by the linear operator $\Delta_D - d$ on the appropriate domain.

For $t > a$, putting $a_0 = 0$ in (2.5), we get

$$\begin{cases} V'(s) = \Delta_D V(s) - dV(s) - b(a_0+s)V(s), \\ V(0) = v(t_0,0,x) = f(x)u(t_0,x). \end{cases} \tag{2.9}$$

Solving (2.9) formally (that is treating $u(t_0, x)$ as a known function) we obtain

$$\begin{aligned} v(t_0+s,s,x) = V(s) &= e^{-\int_0^s b(\sigma)\,d\sigma} e^{s(\Delta_D-d)} v(t_0,0,x) \\ &= e^{-\int_0^s b(\sigma)\,d\sigma} e^{s(\Delta_D-d)} f(x)u(t_0,x). \end{aligned}$$

Hence, for $t > r$ the first equation in (2.1) gives

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta_N u(t,x) - du(t,x) \\ &\quad + \int_0^r b(a) e^{-\int_0^a b(\sigma)\,d\sigma} e^{a(\Delta_D-d)} f(x)u(t-a,x)\,da, \end{aligned} \tag{2.10}$$

which becomes, via $-a = s$, Eq. (1.1). It is a single partial differential equation with delay which we are going to study using the semigroup technique.

If we compare Eq. (1.1) with the model proposed, for instance, by Engel and Nagel in [4, Example VI.6.19] or by Wu in [19, Introduction] (see also [13, Remark 6.2]), we observe that the most relevant difference is that in our case the delay term $u(t+s,x)$ follows $e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\Delta_D-d)}$ which is the evolution family solving the nonautonomous Cauchy problem associated to the operators

$$A(\tau) = \Delta_D - d - b(-\tau), \quad \tau \in [-r,0] \tag{2.11}$$

(see [5,10,14–17]). This takes account of the fact that, in general, pregnant individuals move during the period of gestation and therefore can bear in a place different from that where they were fecundated.

## 3. The population equation as a delay equation with nonautonomous past

### *3.1. The equivalence*

In order to investigate the population equation (1.1) using the semigroup method, we rewrite it as a delay equation with nonautonomous past (see [3,5,6]). Such equations can be written as

$$\text{(NDE)} \quad \begin{cases} \dot u(t) = Bu(t) + \Phi \tilde u_t, & t \geqslant 0, \\ u(0) = y \in X, \\ \tilde u_0 = g \in L^1([-r,0],X), \end{cases}$$

where $(B, D(B))$ is the generator of a strongly continuous semigroup on a Banach space $X$, the delay operator $\Phi : D(\Phi) \subseteq L^1([-r,0],X) \to X$ is a linear operator and $\tilde u_t$ is the modified history function (see [5, Definition 3.2]), i.e., $\tilde u_t : [-r,0] \to X$ is defined as

$$\tilde u_t(\tau) := \begin{cases} \tilde U(\tau, t+\tau)u(t+\tau) & \text{for } t+\tau \geqslant 0, \\ \tilde U(\tau, t+\tau)g(t+\tau) & \text{for } -r \leqslant t+\tau \leqslant 0, \end{cases}$$

for some appropriate evolution family $(\tilde U(t,s))_{t \leqslant s}$. In the definition of the modified history function $\tilde u_t$ two time variables $t$ and $\tau$ appear. The variable $t$ can be interpreted as the *absolute time* and $\tau$ as the *relative time*.

In order to allow the mathematical treatment and preserve biological significance we state the following assumptions and definitions that we will use throughout the paper.

**General assumptions 3.1.** (1) $r = 1$.

(2) *The mortality rate $d$ is a positive constant.*

(3) *The function $\beta(s, x) := f(x)b(-s)$, which depends on the state space variable and on the time, is such that $\beta(\cdot,\cdot)$ is a positive function with $\beta(\cdot, x) \in L^1([-1, 0])$ for each $x \in \Omega$ and $\beta(s,\cdot) \in C(\Omega)$ for each $-1 \leqslant s \leqslant 0$.*

**General definitions 3.2.** (1) *As state space we take $X := L^1(\Omega)$, where $\Omega$ is an open, connected and bounded domain of $\mathbb{R}^N$ with smooth boundary and let $E := L^1([-1, 0], X)$.*

(2) *Let $B := \Delta_N - d$ with domain $D(B) = D(\Delta_N)$ on $X$.*

(3) *Take $\Phi k := \int_{-1}^{0} \beta(s)k(s)\,ds$, where $k \in W^{1,1}([-1, 0], X)$ and $\beta(s) := \beta(s,\cdot)$.*

**Remark 3.3.** (a) The space $L^1(\Omega)$ is the natural state space for the population equation because the $L^1$-norm gives the total population size.

(b) By [1, Proposition 1.9.4], the delay operator $\Phi$ is well-defined.

(c) The data $u_0$ and $g$ are as in (2.4).

About the operator $(B, D(B))$ the following proposition is well known (see, e.g., [4, Chapter VI]).

**Proposition 3.4.** *The operator $(B, D(B))$ is the generator of an analytic contraction semigroup $(S(t))_{t\geqslant 0}$.*

Let $\mathcal{U} := (U(\tau, s))_{-1\leqslant\tau\leqslant s\leqslant 0}$ be the evolution family solving the nonautonomous Cauchy problem associated to the operators $A(\tau) = \Delta_D - d - b(-\tau)$, i.e.,

$$U(\tau, s) := e^{-\int_{-s}^{-\tau} b(\sigma)\,d\sigma} e^{(s-\tau)(\Delta_D - d)} \quad \text{for } -1 \leqslant \tau \leqslant s \leqslant 0, \tag{3.1}$$

and $(\tilde{U}(\tau, s))_{\tau\leqslant s}$ its trivial extension on $\mathbb{R}$ (see, for example, [5, Definition 2.2]). Using (3.1) we can give the following definition.

**Definition 3.5.** The modified history function $\tilde{u}_t$ is

$$\begin{aligned}
\tilde{u}_t(s) &:= \begin{cases} \tilde{U}(s, s+t)g(s+t), & -1 \leqslant s+t \leqslant 0, \\ \tilde{U}(s, s+t)u(s+t), & s+t \geqslant 0, \end{cases} \\
&= \begin{cases} U(s, s+t)g(s+t), & -1 \leqslant s+t \leqslant 0, \\ U(s, 0)u(s+t), & s+t \geqslant 0, \end{cases} \\
&= \begin{cases} e^{-\int_{-s-t}^{-s} b(\sigma)\,d\sigma} e^{t(\Delta_D - d)} g(t+s), & -1 \leqslant s+t \leqslant 0, \\ e^{-\int_{0}^{-s} b(\sigma)\,d\sigma} e^{-s(\Delta_D - d)} u(t+s), & s+t \geqslant 0. \end{cases}
\end{aligned}$$

Using the previous definitions and setting $y := u_0(x)$, Eq. (1.1) becomes a delay equation with nonautonomous past (NDE).

We call a function $u : [-1,+\infty) \to X$ a *classical solution* of (NDE) if

(i) $u \in C([-1,+\infty), X) \cap C^1(\mathbb{R}_+, X)$,
(ii) $u(t) \in D(B)$, $\tilde{u}_t \in D(\Phi)$, $t \geqslant 0$,
(iii) $u$ satisfies (NDE) for all $t \geqslant 0$.

### 3.2. Well-posedness

Now we want to find a solution of (1.1). By the previous section, this is equivalent to study the well-posedness of (NDE).

**Definition 3.6.** We call (NDE) *well-posed* if

(i) for every $\binom{y}{g}$ in a dense subspace $\mathcal{S} \subseteq X \times L^1([-1,0], X)$ there is a unique (classical) solution $u(y,g,\cdot)$ of (NDE) and
(ii) the solutions depend continuously on the initial values, i.e., if a sequence $\binom{y_n}{g_n}$ in $\mathcal{S}$ converges to $\binom{y}{g} \in \mathcal{S}$, then $u(y_n, g_n, t)$ converges to $u(y,g,t)$ uniformly for $t$ in compact intervals.

As in [5, Theorem 3.5] one can prove that the well-posedness of (NDE) is equivalent to the well-posedness of the following abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \dot{\mathcal{W}}(t) = \mathcal{C}\mathcal{W}(t), \\ \mathcal{W}(0) = \binom{y}{g} \end{cases}$$

on the product space $\mathcal{E} = X \times L^1([-1,0], X)$, where the operator $\mathcal{C}$ is the matrix

$$\mathcal{C} := \begin{pmatrix} B & \Phi \\ 0 & G \end{pmatrix}, \tag{3.2}$$

with domain

$$D(\mathcal{C}) := \left\{ \begin{pmatrix} y \\ g \end{pmatrix} \in D(B) \times D(G) \colon g(0) = y \right\}. \tag{3.3}$$

Here the operator $(G, D(G))$ is the closure of

$$(Ag)(s) := g'(s) + \big(\Delta_D - d - b(-s)\big) g(s),$$

for $g \in D := \{g \in W^{1,1}([-1,0], X) \colon g(0) \in D(\Delta_N),\ g(s) \in D(\Delta_D),\ s \mapsto (\Delta_D - d - b(-s))g(s) \in L^1([-1,0], X)\}$ (see [7, Proposition 3.1 and Lemma 3.2]), while the operators $(B, D(B))$ and $\Phi$ are as in General definitions 3.2.

**Remark 3.7.** Observe that, in fact, the results proved in [5–7] for $L^p(\mathbb{R}_-, X)$ hold for $L^p([-r,0], X)$, for all $r > 0$, as well.

For the operator $(\mathcal{C}, D(\mathcal{C}))$ the following theorem holds.

**Theorem 3.8.** *If General assumptions* (3.1) *hold, then the operator* $(\mathcal{C}, D(\mathcal{C}))$ *defined in* (3.2) *generates a positive semigroup* $(\mathcal{T}(t))_{t\geqslant 0}$ *on the product space* $\mathcal{E}$.

**Proof.** Rewrite the operator $\mathcal{C}$ in the form

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{F} := \begin{pmatrix} B & 0 \\ 0 & G \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

with domain $D(\mathcal{C}_0) = D(\mathcal{C})$ and $\mathcal{F} \in \mathcal{L}(D(\mathcal{C}_0), \mathcal{E})$. As in [5, Proposition 4.2], we can prove that $(\mathcal{C}_0, D(\mathcal{C}_0))$ is a generator.

Now, let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $q(t) := \|b\|_1 t$. Then $\lim_{t\to 0} q(t) = 0$. As in [5, Example 4.6] one can prove the following Miyadera–Voigt inequality (see also [4, Theorem III.3.14], [9] or [18])

$$\int_0^t \left\| \int_{-1}^0 \beta(s)\big[(S_r y)(s) + \big(T_0(r)g\big)(s)\big]\,ds \right\| dr \leqslant q(t) \left\| \begin{pmatrix} y \\ g \end{pmatrix} \right\| \tag{3.4}$$

for all $\binom{y}{g} \in D(\mathcal{C})$, where the function $\beta$ is as in General assumptions (3.1). Here,

$$\begin{aligned}(S_t y)(s) &:= \begin{cases} U(s,0)S(t+s)y, & t+s \geqslant 0, \\ 0, & \text{elsewhere,} \end{cases} \\ &= \begin{cases} e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\Delta_D - d)} S(t+s)y, & t+s \geqslant 0, \\ 0, & \text{elsewhere,} \end{cases}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(T_0(t)g)(s) &:= \begin{cases} 0, & s+t > 0, \\ U(s, s+t)g(t+s), & -1 \leqslant s+t \leqslant 0, \end{cases} \\ &= \begin{cases} 0, & s+t > 0, \\ e^{-\int_{-s-t}^{-s} b(\sigma)\,d\sigma} e^{t(\Delta_D - d)} g(t+s), & -1 \leqslant s+t \leqslant 0, \end{cases}\end{aligned} \tag{3.6}$$

for $g \in L^1([-1,0], X)$. Recall that $(S(t))_{t\geqslant 0}$ is the semigroup given in Proposition 3.4. By the perturbation theorem of Miyadera and Voigt (see, e.g., [4, Corollary III.3.16]), the operator $(\mathcal{C}, D(\mathcal{C}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on the product space $\mathcal{E}$. Moreover, since $e^{t\Delta_D}$ is positive, $(S(t))_{t\geqslant 0}$ is positive too and as in [2, Lemma 2.45], the positivity of $(\mathcal{T}(t))_{t\geqslant 0}$ follows as well. □

As an immediate consequence of [5, Theorem 3.5] one has the next theorem.

**Theorem 3.9.** *If General assumptions* (3.1) *hold, then the delay equation with nonautonomous past* (*NDE*) *is well-posed and the solution* $u$, *for* $\binom{y}{g} \in D(\mathcal{C})$, *is given by*

$$u(t) = \begin{cases} \pi_1\big(\mathcal{T}(t)\binom{y}{g}\big), & t \geqslant 0, \\ g(t), & -1 \leqslant t \leqslant 0, \end{cases}$$

*where* $\pi_1$ *is the projection onto the first component of* $\mathcal{E}$.

## 4. The stability

This section is devoted to study the stability of the solutions of (1.1). Namely, we look for conditions such that the solutions decay exponentially, i.e., $\omega_0(\mathcal{T}(\cdot)) < 0$. This is important, for example, if $u$ denotes the population density of a virus. The case of $\omega_0(\mathcal{T}(\cdot)) \geqslant 0$ is studied by Nickel and Rhandi in [11]. In order to analyze the stability, we will use the following result, which can be proved as in [6, Theorem 4.1].

**Theorem 4.1.** *Assume that the operator $(B, D(B))$ generates an immediately norm continuous semigroup $(S(t))_{t\geqslant 0}$. Then the growth bound of $\mathcal{T}$ is given by*

$$\omega_0(\mathcal{T}(\cdot)) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}.$$

So we have to look at the growth bound of the evolution family $\mathcal{U}$, $\omega_0(\mathcal{U})$, and the spectral bound of $\mathcal{C}$, $s(\mathcal{C})$.

Let $(T(t))_{t\geqslant 0}$ be the semigroup generated by $(\Delta_D, D(\Delta_D))$. By the definition of $\mathcal{U}$ (see (3.1)), $\omega_0(\mathcal{U}) \leqslant \omega_0(T(\cdot))$, and, observing that $\omega_0(T(\cdot)) < 0$, the next result arises immediately.

**Proposition 4.2.** *For the growth bound of the evolution family $\mathcal{U}$ it holds*

$$\omega_0(\mathcal{U}) < 0.$$

To estimate the spectral bound of $\mathcal{C}$, using the positivity of $(\mathcal{T}(t))_{t\geqslant 0}$ (see Theorem 3.8), the following lemma is helpful.

**Lemma 4.3.** *For $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$, we have*

$$s(\mathcal{C}) < \lambda \quad \text{if and only if} \quad s(B + \Phi\epsilon_\lambda) < \lambda,$$

*where the bounded operators $\epsilon_\lambda : X \to E := L^1([-1, 0], X)$ are defined by*

$$(\epsilon_\lambda x)(s) := e^{\lambda s} U(s, 0)x, \tag{4.1}$$

*for $s \in [-1, 0]$, $x \in X$ and $\lambda$ as before.*

The proof can be obtained rewriting the one given in [6, Lemma 3.1]. As a straightforward consequence, the next proposition holds.

**Proposition 4.4.** *For $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$, we have that*

$$s(\mathcal{C}) < \lambda \quad \Leftrightarrow \quad \int_{-r}^{0} \beta(s) e^{s(\lambda - \lambda_0 + d)} e^{-\int_0^{-s} b(\sigma)\, d\sigma}\, ds < \lambda + d, \tag{4.2}$$

*where $\lambda_0$ is largest eigenvalue of $\Delta_D$.*

**Proof.** By definition, one has

$$B + \Phi\epsilon_\lambda = \Delta_N - d + \int_{-r}^{0} \beta(s) e^{\lambda s} e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\Delta_D - d)}\,ds,$$

where $\epsilon_\lambda$ is the function defined in the previous lemma. Using the spectral theorem for self-adjoint operators, this implies

$$s(B + \Phi\epsilon_\lambda) = \gamma_0 - d + \int_{-r}^{0} \beta(s) e^{\lambda s} e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\lambda_0 - d)}\,ds,$$

where $s(B + \Phi\epsilon_\lambda)$ is the spectral bound of the operator $B + \Phi\epsilon_\lambda$, and $\gamma_0$ and $\lambda_0$ are the largest eigenvalues of $\Delta_N$ and $\Delta_D$, respectively. Since $\gamma_0 = 0$, it follows that

$$s(B + \Phi\epsilon_\lambda) = -d + \int_{-r}^{0} \beta(s) e^{\lambda s} e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\lambda_0 - d)}\,ds.$$

From Lemma 4.3, we can conclude that

$$s(\mathcal{C}) < \lambda \quad \Leftrightarrow \quad \int_{-r}^{0} \beta(s) e^{\lambda s} e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\lambda_0 - d)}\,ds < \lambda + d. \qquad \square$$

**Corollary 4.5.** *For the spectral bound of the operator* $(\mathcal{C}, D(\mathcal{C}))$ *the following property holds*:

$$\begin{aligned} s(\mathcal{C}) < 0 \quad &\Leftrightarrow \quad s(B + \Phi\epsilon_0) < 0 \\ &\Leftrightarrow \quad \int_{-r}^{0} \beta(s) e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\lambda_0 - d)}\,ds < d. \end{aligned}$$

Using Theorem 4.1, Proposition 4.2 and the previous corollary we can obtain conditions under which the semigroup $(\mathcal{T}(t))_{t \geqslant 0}$ is uniformly exponentially stable.

**Proposition 4.6.** *Assume that* $\int_{-r}^{0} \beta(s) e^{-\int_0^{-s} b(\sigma)\,d\sigma} e^{-s(\lambda_0 - d)}\,ds < d$. *Then*

$$\omega_0\big(\mathcal{T}(\cdot)\big) < 0.$$

**Corollary 4.7.** *Under the assumption of the previous proposition the solutions of* (1.1) *decay exponentially.*

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## References


[1] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, in: Monographs in Mathematics, Vol. 96, Birkhäuser, 2001.

[2] A. Bátkai, S. Piazzera, Semigroups for delay equations in $L^p$-phase spaces, Preprint, 2000.

[3] S. Brendle, R. Nagel, Partial functional differential equations with nonautonomous past, Discrete Contin. Dynam. Systems 8 (2002) 953–966.

[4] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, 2000.

[5] G. Fragnelli, G. Nickel, Partial functional differential equations with nonautonomous past in $L^p$-phase spaces, Differential Integral Equations 16 (2003) 327–348.

[6] G. Fragnelli, A spectral mapping theorem for semigroups solving PDEs with nonautonomous past in $L^p$-phase spaces, Abstract Appl. Anal. 8 (16) (2003) 933–951.

[7] G. Fragnelli, Classical solution for PDEs with nonautonomous past in $L^p$-spaces, Bull. Belg. Math. Soc., in press.

[8] P. Marcati, M.A. Pozio, Global asymptotic stability for a vector disease model with spatial spread, J. Math. Biol. 9 (1980) 179–187.

[9] I. Miyadera, On perturbation theory for semigroups of operators, Tohoku Math. J. 18 (1966) 299–310.

[10] G. Nickel, Evolution semigroups for nonautonomous Cauchy problems, Abstract Appl. Anal. 2 (1997) 73–95.

[11] G. Nickel, A. Rhandi, Positivity and stability of delay equations with nonautonomous past, in press.

[12] M.A. Pozio, Behaviour of solutions for some abstract functional differential equations and application to predator-prey dynamics, Nonlinear Anal. 4 (1980) 917–938.

[13] M.A. Pozio, Some conditions for global asymptotic stability of equilibria of integrodifferential equations, J. Math. Anal. Appl. 95 (1983) 501–527.

[14] R. Schnaubelt, A sufficient condition for exponentially dichotomy of parabolic evolution equations, in: Proc. 6th International Conference on Evolution Equations and Semigroups, in press.

[15] R. Schnaubelt, Exponential bounds and hyperbolicity of evolution families, Ph.D. thesis, Tübingen, 1996.

[16] R. Schnaubelt, Exponential dichotomy of non-autonomous evolution equations, Habilitationsschrift, Tübingen, 1999.

[17] R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations, Forum Math. 11 (1999) 543–566.

[18] J. Voigt, On the perturbation theory for strongly continuous semigroups, Math. Ann. 229 (1977) 163–171.

[19] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, 1996.